

Deformation equivalence classes of complex surfaces with the first Betti number one, and the second Betti number zero

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Abstract

We will prove that the number of deformation equivalence classes of surfaces homotopy equivalent to a smooth, closed 4-manifold is finite, if the first Betti number is equal to one, and the second Betti number is equal to zero.

Introduction

Recently, the study of compact complex surfaces has made many outstanding progresses. One of the topics in this area is to investigate the discrepancy between homotopy class, homeomorphism class, diffeomorphism class and deformation equivalence classes of surfaces. In this paper we ask whether the number of deformation equivalence classes of surfaces homotopy equivalent (or diffeomorphic) to a smooth closed 4-manifold is finite or not. For this problem, the celebrated Yau's result [12] states that any surface homotopy equivalent to \mathbb{CP}^2 is biholomorphic to \mathbb{CP}^2 , hence there exists only one deformation equivalence class of surfaces homotopy equivalent to \mathbb{CP}^2 . According to [3], the number of deformation equivalence classes of surface diffeomorphic to M is finite, if M is a smooth, closed 4-manifold M with $b_1(M) \neq 1$. It is also known that the number of deformation equivalence classes of surfaces homotopy equivalent to M is finite if and only if M is not homotopy equivalent to an elliptic surface whose fundamental group is finite cyclic. Our result is an affirmative answer for M with $b_1(M) = 1$ and $b_2(M) = 0$.

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Main Theorem Let M be a smooth, closed 4-manifold with $b_1(M) = 1$ and $b_2(M) = 0$. Then the number of deformation equivalence classes of surfaces homotopy equivalent to M is finite.

Our strategy is as follows. Let S be a compact surface with $b_1(S) = 1$ and $b_2(S) = 0$. By the Enriques-Kodaira classification [1] and the results of Bogomolov [2], and Teleman [10], S is biholomorphic to an elliptic surface, a Hopf surface, or an Inoue surface. We first assume that S is an Inoue surface. We show that S cannot deform to surfaces other than Inoue surfaces by comparing the fundamental groups. The fundamental group also distinguishes all the Inoue surfaces homotopy equivalent to S . We show that the number of deformation equivalence classes is finite by constructing biholomorphic maps explicitly. Then, we will prove the theorem for elliptic surfaces and Hopf surfaces using the results of [3].

We remark that the classification of surfaces with $b_1 = 1, b_2 > 0$ and Kodaira dimension $-\infty$, remains open. It is conjectured that S contains a global spherical shell [11] if S is minimal. If this conjecture is proved affirmatively, [3] and [8] implies that any surface with $b_1 = 1$ and $b_2 > 0$ is deformation equivalent to a surface with $b_1 = 1, b_2 = 0$ blown up finitely times. Hence, by our main theorem, it follows that the number of deformation equivalence classes of surfaces diffeomorphic to a closed 4-manifold is finite.

1 Inoue surfaces of type S^0

In [6], Inoue constructed surfaces with $b_1 = 1, b_2 = 0$. We follow [5] to divide them into three types; type S^0, S^+ , and S^- . We begin by recalling the definition of Inoue surfaces of type S^0 .

Let $M = (m_{ij}) \in SL(3, \mathbb{Z})$ be a matrix with eigenvalues $\alpha > 1, \beta, \bar{\beta} \in \mathbb{C} \setminus \mathbb{R}$. We choose eigenvectors $\mathbf{a} = {}^T(a_1, a_2, a_3) \in \mathbb{R}^3$ and $\mathbf{b} = {}^T(b_1, b_2, b_3) \in \mathbb{C}^3$ of M corresponding to α and β respectively. Define G_M to be the subgroup of $\text{Aut}(\mathbb{H} \times \mathbb{C})$ generated by

$$g_0 : (w, z) \mapsto (\alpha w, \beta z), \quad (1)$$

$$g_i : (w, z) \mapsto (w + a_i, z + b_i), \quad (2)$$

where $i = 1, 2, 3$, and \mathbb{H} is the upper half complex plane. The quotient surface $S_{M, \mathbf{a}, \mathbf{b}}^0 = \mathbb{H} \times \mathbb{C} / G_M$ is called an Inoue surface of type S^0 .

Recall that two complex manifolds S_1 and S_2 are deformation equivalent if

there exists connected complex manifolds χ and B , a smooth proper holomorphic map $\pi : \chi \rightarrow B$, and two points $t_1, t_2 \in B$ such that $\pi^{-1}(t_i)$ is biholomorphic to S_i . In [7], Inoue showed that if S is a surface whose fundamental group is isomorphic to $\pi_1(S_{M,\mathbf{a},\mathbf{b}}^0)$, then S is biholomorphic either to $S_{M,\mathbf{a},\mathbf{b}}^0$ or $S_{M,\mathbf{a},\bar{\mathbf{b}}}^0$. He also showed that $S_{M,\mathbf{a},\mathbf{b}}^0$ and $S_{M,\mathbf{a},\bar{\mathbf{b}}}^0$ are not deformation equivalent. Hence, the number of deformation equivalence classes of surfaces homotopy equivalent to S is exactly two.

2 Inoue surfaces of type S^+

Let $N = (n_{ij}) \in SL(2, \mathbb{Z})$ be a matrix with eigenvalues $\alpha > 1, 1/\alpha$. We choose eigenvectors $\mathbf{a} = {}^T(a_1, a_2) \in \mathbb{R}^2$ and $\mathbf{b} = {}^T(b_1, b_2) \in \mathbb{R}^2$ of N corresponding to α and $1/\alpha$ respectively. Fix $t \in \mathbb{C}$ and integers p, q, r , where $r \neq 0$. Let $\theta = \det(\mathbf{a}, \mathbf{b})$. Define $c_1, c_2 \in \mathbb{R}$ to be solution of the following equation

$$(N - I) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} - \frac{\theta}{r} \begin{pmatrix} p \\ q \end{pmatrix} = 0,$$

where $e_i = \frac{1}{2}n_{i1}(n_{i1} - 1)a_1b_1 + \frac{1}{2}n_{i2}(n_{i2} - 1)a_2b_2 + n_{i1}n_{i2}b_1a_2$, and $i = 1, 2$. Let $G_{N,p,q,r}^+$ be the subgroup of $\text{Aut}(\mathbb{H} \times \mathbb{C})$ generated by

$$\begin{aligned} g_0 : (w, z) &\mapsto (\alpha w, z + t), \\ g_i : (w, z) &\mapsto (z + a_i, w + b_i z + c_i), \\ g_3 : (w, z) &\mapsto (z, w - \theta/r), \end{aligned}$$

where $i = 1, 2$. The quotient surface $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+ = \mathbb{H} \times \mathbb{C} / G_{N,p,q,r}^+$ is called an Inoue surface of type S^+ .

We have the following relations:

$$g_0 g_1 g_0^{-1} = g_1^{n_{11}} g_2^{n_{12}} g_3^p, \quad (3)$$

$$g_0 g_2 g_0^{-1} = g_1^{n_{21}} g_2^{n_{22}} g_3^q, \quad (4)$$

$$g_1 g_2 g_1^{-1} g_2^{-1} = g_3^r, \quad (5)$$

$$g_i g_3 = g_3 g_i, \quad (6)$$

where $i = 0, 1, 2, 3$. Any element $g \in G_{N,p,q,r}^+$ can be written uniquely as a product $g = \prod_{i=0}^3 g_i^{n_i}$, where n_0, n_1, n_2, n_3 are integers. Thus, $G_{N,p,q,r}^+$ has a presentation in terms of the generators g_0, g_1, g_2, g_3 , with relations (3) - (6). The center of $G_{N,p,q,r}^+$ is an infinite cyclic group generated by g_3 .

Notation Let G be a group. We denote by Γ_G the normal subgroup

$$\{g \in G \mid g \bmod [G, G] \text{ is of finite order}\}, \quad (7)$$

where $[G, G]$ is the commutator subgroup.

The subgroup $\Gamma_{G_{N,p,q,r}^+}$ of $G_{N,p,q,r}^+$ has the following properties:

- (i) $\Gamma_{G_{N,p,q,r}^+}$ is generated by g_1, g_2, g_3 ,
- (ii) the center of $\Gamma_{G_{N,p,q,r}^+}$ is an infinite cyclic group generated by g_3 ,
- (iii) the quotient group of $\Gamma_{G_{N,p,q,r}^+}$ by its center is a free abelian group of rank two generated by the classes of g_1 and g_2 ,
- (iv) the quotient group $G_{N,p,q,r}^+ / \Gamma_{G_{N,p,q,r}^+}$ is an infinite cyclic group generated by the class of g_0 .

LEMMA 2.1. *Inoue surfaces $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+$ and $S_{N',p',q',r',t',\mathbf{a}',\mathbf{b}'}^+$ constructed above are homotopy equivalent if and only if there exists $K \in GL(2, \mathbb{Z})$, $\delta, \epsilon \in \{1, -1\}$ such that*

$$r' = \delta \det K r, \quad (8)$$

$$N' = K N^\epsilon K^{-1}, \quad (9)$$

$$\delta \begin{pmatrix} p' \\ q' \end{pmatrix} - \epsilon K \begin{pmatrix} p \\ q \end{pmatrix} \in r\mathbb{Z}^2 + (N' - I)\mathbb{Z}^2. \quad (10)$$

Proof Assume that $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+$ and $S_{N',p',q',r',t',\mathbf{a}',\mathbf{b}'}^+$ are homotopy equivalent. Then, there exists an isomorphism $\varphi : G_{N',p',q',r'}^+ \rightarrow G_{N,p,q,r}^+$. By property (ii), $\varphi(g'_3) = g_3^\delta$ for some $\delta \in \{1, -1\}$. From property (i), there exists integers k_{i1}, k_{i2}, k_{i3} such that

$$\varphi(g'_i) = g_1^{k_{i1}} g_2^{k_{i2}} g_3^{k_{i3}}, \quad (11)$$

where $i = 1, 2$. We infer from property (iii) that if we let $K = (k_{ij})_{i,j=1,2}$, then $K \in GL(2, \mathbb{Z})$. Property (iv) implies that there exists $\epsilon \in \{1, -1\}$, and integers l_1, l_2, l_3 such that

$$\varphi(g'_0) = g_0^\epsilon g_1^{l_1} g_2^{l_2} g_3^{l_3}. \quad (12)$$

By relation (5), we obtain (8). For the sake of simplicity, let $n'_{13} = p', n'_{23} = q'$. Suppose $\epsilon = 1$. By (11) and (12), we have

$$\begin{aligned}\varphi(g'_0 g'_i g'^{-1}_0) &= g_1^{(k_{i1}, k_{i2})^T (n_{11}, n_{21})} g_2^{(k_{i1}, k_{i2})^T (n_{12}, n_{22})} g_3^{k_{i3} + (k_{i1}, k_{i2})^T (p, q) + r(k_{i1}, k_{i2})^T (-l_2, l_1) + ra}, \\ \varphi(g'^{n'_{i1}}_1 g'^{n'_{i2}}_2 g'^{n'_{i3}}_3) &= g_1^{(n'_{i1}, n'_{i2})^T (k_{11}, k_{21})} g_2^{(n'_{i1}, n'_{i2})^T (k_{12}, k_{22})} g_3^{(n'_{i1}, n'_{i2})^T (k_{13}, k_{23}) + \delta n'_{i3} + rb},\end{aligned}$$

where $i = 1, 2$ and a, b are integers which depends only on K, N , and N' . Thus,

$$\begin{aligned}KNK^{-1} &= N', \\ \delta \begin{pmatrix} p' \\ q' \end{pmatrix} - K \begin{pmatrix} p \\ q \end{pmatrix} &\in (N' - I)\mathbb{Z}^2 + r\mathbb{Z}^2.\end{aligned}$$

Similarly, if $\epsilon = -1$, we obtain $KN^{-1}K^{-1} = N'$ and

$$\delta \begin{pmatrix} p' \\ q' \end{pmatrix} + K \begin{pmatrix} p \\ q \end{pmatrix} \in (N' - I)\mathbb{Z}^2 + r\mathbb{Z}^2.$$

Hence, we get (9) and (10).

Next we show the converse. Recall that the diffeomorphism type of solvmanifolds is determined by the fundamental group. Since $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+$ and $S_{N',p',q',r',t',\mathbf{a}',\mathbf{b}'}^+$ are both solvmanifolds [5], it suffices to show that $G_{N',p',q',r'}^+$ and $G_{N,p,q,r}^+$ are isomorphic. By (10), there exists $k_{13}, k_{23}, u, v \in \mathbb{Z}$ such that

$$\delta \begin{pmatrix} p' \\ q' \end{pmatrix} - \epsilon K \begin{pmatrix} p \\ q \end{pmatrix} = (N' - I) \begin{pmatrix} k_{13} \\ k_{23} \end{pmatrix} + r \begin{pmatrix} u \\ v \end{pmatrix}.$$

Suppose $\epsilon = 1$. Let l_1, l_2 be integers which satisfy the following equation:

$$\begin{pmatrix} -l_2 \\ l_1 \end{pmatrix} = K^{-1} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} a \\ b \end{pmatrix}.$$

The desired isomorphism $\varphi : G_{N',p',q',r'}^+ \rightarrow G_{N,p,q,r}^+$ is given by

$$\begin{aligned}\varphi(g'_0) &= g_0^\epsilon g_1^{l_1} g_2^{l_2}, \\ \varphi(g'_i) &= g_1^{k_{i1}} g_2^{k_{i2}} g_3^{k_{i3}}, \\ \varphi(g'_3) &= g_3^\delta,\end{aligned}$$

where $i = 1, 2$. The case $\epsilon = -1$ can be proved similarly. \square

In [4], Fujimoto and Nakayama introduced the group Γ_r to construct a surjective holomorphic map from $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+$ to itself. We apply their method to construct a biholomorphic map between two Inoue surfaces of type S^+ with

different parameters. However, we will assume that $r = r'$, which is enough to show our main theorem. We first review its definition. Let $\Gamma_r = \mathbb{Z}^2 \times \mathbb{Z}[r/2]$, where $\mathbb{Z}[r/2] = \mathbb{Z}$ if r is even, and $\frac{1}{2}\mathbb{Z}$ if r is odd. The group law of Γ_r is defined as follows:

$$(\zeta, y)(\zeta', y') = (\zeta + \zeta', y + y' + \frac{r}{2} \det(\zeta, \zeta')), \quad (13)$$

where $\zeta, \zeta' \in \mathbb{Z}^2$ are row vectors, and $y, y' \in \mathbb{Z}[r/2]$. By (8.4) of [4], we have

$$(N - I)\mathbf{c} = \frac{\theta}{r}\mathbf{p}, \quad (14)$$

where $\mathbf{c} = {}^T(c_1 - a_1 b_1/2, c_2 - a_2 b_2/2)$, and $\mathbf{p} = {}^T(p + (r/2)n_{11}n_{12}, q + (r/2)n_{21}n_{22})$. Let $\mu : \Gamma_{G_{N,p,q,r}^+} \rightarrow \Gamma_r$ be an injective homomorphism given by

$$g_1^{l_1} g_2^{l_2} g_3^{l_3} \mapsto ((l_1, l_2), l_3 + l_1 l_2 \frac{r}{2}). \quad (15)$$

The group Γ_r acts on $\mathbb{H} \times \mathbb{C}$ as follows:

$$(w, z) \mapsto (\zeta, y)(w, z) = \left(w + \zeta \mathbf{a}, z + (\zeta \mathbf{b})w + \zeta \mathbf{c} - \frac{\theta}{r}y + \frac{1}{2}(\zeta \mathbf{a})(\zeta \mathbf{b}) \right). \quad (16)$$

The action above factors through Γ_r by μ .

Let us review the endomorphisms of Γ_r . According to Lemma 8.4 of [4], an endomorphism φ of Γ_r can be written as

$$\varphi(\zeta, y) = (\zeta K, \zeta \mathbf{v} + (\det K)y),$$

for some $K = (k_{ij}) \in M_2(\mathbb{Z})$ and $\mathbf{v} = {}^T(v_1, v_2) \in \mathbb{Z}[r/2]^2$. Furthermore, $\text{End}(\Gamma_r)$ is anti-isomorphic to the semi-group $M_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ via the map

$$\text{End}(\Gamma_r) \rightarrow M_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2 : \varphi \mapsto (K, \mathbf{v}), \quad (17)$$

where the multiplicative structure on $M_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ is defined by

$$(K_1, \mathbf{v}_1)(K_2, \mathbf{v}_2) = (K_1 K_2, K_1 \mathbf{v}_2 + (\det K_2) \mathbf{v}_1).$$

Throughout this paper, we identify $\text{End}(\Gamma_r)$ with $M_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ by (17).

LEMMA 2.2. *Assume that $r = r'$.*

- (1) *Any homomorphism $\Gamma_{G_{N',p',q',r'}^+} \rightarrow \Gamma_{G_{N,p,q,r}^+}$ lifts to a unique endomorphism of Γ_r .*
- (2) *The pair $(K, \mathbf{v}) \in M_2(\mathbb{Z}) \times \mathbb{Z}[r/2]^2$ is induced from a homomorphism $\Gamma_{G_{N',p',q',r'}^+} \rightarrow \Gamma_{G_{N,p,q,r}^+}$ if and only if $v_1 - (r/2)k_{11}k_{12}, v_2 - (r/2)k_{21}k_{22} \in \mathbb{Z}$ where $K = (k_{ij})$, and $\mathbf{v} = {}^T(v_1, v_2)$.*

- (3) The lift of the automorphism $\nu : \gamma \mapsto g_0 \gamma g_0^{-1}$ of $\Gamma_{G_{N,p,q,r}^+}$ corresponds to (N, \mathbf{p}) .
- (4) Let $\rho : G_{N',p',q',r'}^+ \rightarrow G_{N,p,q,r}^+$ be an isomorphism such that $\rho(g'_0) = g_0 g_1^{l_1} g_2^{l_2}$ for some integers l_1, l_2 . Denote $(K, \mathbf{v}) \in GL(2, \mathbb{Z}) \times \mathbb{Z}[r/2]^2$ the lift of $\rho|_{\Gamma_{G_{N',p',q',r'}^+}}$. Then (K, \mathbf{v}) satisfies the following conditions:

$$KN = N'K, \quad (N' - I)\mathbf{v} = K\mathbf{p} - (\det K)\mathbf{p}' + Kr \begin{pmatrix} -l_2 \\ l_1 \end{pmatrix}, \quad (18)$$

$$v_i - (r/2)k_{i1}k_{i2} \in \mathbb{Z}, \quad (19)$$

where $\mathbf{v} = {}^T(v_1, v_2)$. Conversely, assume that $(K, \mathbf{v}) \in GL(2, \mathbb{Z}) \times \mathbb{Z}[r/2]^2$ satisfies (18) and (19) for some integers l_1 and l_2 . Let $\varphi : \Gamma_{G_{N',p',q',r'}^+} \rightarrow \Gamma_{G_{N,p,q,r}^+}$ be a isomorphism whose lift corresponds to (K, \mathbf{v}) . Then, the map

$$\rho : G_{N',p',q',r'}^+ \rightarrow G_{N,p,q,r}^+ : g'^{l_0} \gamma \mapsto (g_0 g_1^{l_1} g_2^{l_2})^{l_0} \varphi(\gamma),$$

is an isomorphism, where $\gamma \in \Gamma_{G_{N',p',q',r'}^+}$.

Proof Let $\varphi : \Gamma_{G_{N',p',q',r'}^+} \rightarrow \Gamma_{G_{N,p,q,r}^+}$ be a homomorphism. Then $\varphi(g'_i) = g_1^{k_{i1}} g_2^{k_{i2}} g_3^{k_{i3}}$ for some integers k_{ij} , where $i = 1, 2$ and $j = 1, 2, 3$. If we let $K = (k_{ij})_{i,j=1,2}$, we see that the endomorphism of Γ_r corresponding to

$$\left(K, \begin{pmatrix} k_{13} + \frac{r}{2}k_{11}k_{12} \\ k_{23} + \frac{r}{2}k_{21}k_{22} \end{pmatrix} \right) \in GL(2, \mathbb{Z}) \times \mathbb{Z}[r/2]^2$$

is the lift of φ . This shows (1). It is obvious that $((l_1, l_2), \lambda) \in \Gamma_r$ is in the image of μ if and only if $\lambda - (r/2)l_1l_2 \in \mathbb{Z}$, hence we have (2). (3) follows from relations (3) and (4). For (4), let ι be the automorphism of $\Gamma_{G_{N,p,q,r}^+}$ given by the conjugation of $\eta = g_1^{l_1} g_2^{l_2}$. We define the automorphism ν' of $\Gamma_{G_{N',p',q',r'}^+}$ similarly to ν . Let $\varphi = \rho|_{\Gamma_{G_{N',p',q',r'}^+}}$. Then, $\rho(g'_0 \gamma g'^{-1}_0) = g_0 \eta \rho(\gamma) \eta^{-1} g_0^{-1}$ implies

$$\varphi \circ \nu' = \nu \circ \iota \circ \varphi. \quad (20)$$

Since the lift of ι corresponds to $(I, r^T(-l_2, l_1))$, we obtain

$$(N', \mathbf{p}')(K, \mathbf{v}) = (K, \mathbf{v})(I, r^T(-l_2, l_1))(N, \mathbf{p}),$$

which is equivalent to (18).

We now show the converse. By assumption, we have (20). It follows that ρ is an isomorphism by calculation. \square

LEMMA 2.3. Assume that $r = r'$ and that there exists $K \in GL(2, \mathbb{Z}), c > 0, u, v \in \mathbb{Z}$ such that

$$KNK^{-1} = N', \quad (21)$$

$$c\mathbf{a}' = K\mathbf{a}, f\mathbf{b}' = cK\mathbf{b}, \quad (22)$$

$$ft' - t = \frac{-\alpha}{\alpha - 1}(\eta\mathbf{a})(\eta\mathbf{b}) + \eta c + \frac{1}{2}(\eta\mathbf{a})(\eta\mathbf{b}) - \frac{\theta}{r}l_1l_2, \quad (23)$$

$$\eta \in \mathbb{Z}^2,$$

where $f = \frac{\det K \theta}{\theta'}$, and $\eta = (l_1, l_2)$ is a solution of the following equation:

$$(N' - I) \begin{pmatrix} u + \frac{r}{2}k_{11}k_{12} \\ v + \frac{r}{2}k_{21}k_{22} \end{pmatrix} + (\det K)\mathbf{p}' = K\mathbf{p} + Kr \begin{pmatrix} -l_2 \\ l_1 \end{pmatrix}. \quad (24)$$

Then $S_{N', p, q, r, t, \mathbf{a}, \mathbf{b}}^+$ and $S_{N', p', q', r', t', \mathbf{a}', \mathbf{b}'}^+$ are biholomorphic.

Proof Let $\varphi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{H} \times \mathbb{C} : (w, z) \mapsto (cw + d, ew + fz)$, where

$$d = -\frac{\alpha}{\alpha - 1}(\eta\mathbf{a}), e = \frac{c}{\alpha - 1}(\eta\mathbf{b}). \quad (25)$$

Set $\mathbf{v} = \begin{pmatrix} u + \frac{r}{2}k_{11}k_{12} \\ v + \frac{r}{2}k_{21}k_{22} \end{pmatrix}$. By (21) and (24), there exist an isomorphism $\rho : G_{N', p', q', r'}^+ \rightarrow G_{N, p, q, r}^+$ defined in Lemma 2.2. It suffices to show that $\varphi \circ g'_0 \circ \varphi^{-1} = g_0 g_1^{l_1} g_2^{l_2}$ and $\varphi \circ g' \circ \varphi^{-1} = \rho(g')$ for any $g' \in \Gamma_{G_{N', p', q', r'}^+}$. Note that

$$g_0 g_1^{l_1} g_2^{l_2}(w, z) = g_0(\mu(g_1 g_2)(w, z)).$$

Hence we obtain $\varphi \circ g'_0 \circ \varphi^{-1} = g_0 g_1^{l_1} g_2^{l_2}$ by (23) and (25).

Now, let Γ'_r be a copy of Γ_r . We define the homomorphism $\mu' : \Gamma_{G_{N', p', q', r'}^+} \rightarrow \Gamma'_r$ and the action of Γ'_r on $\mathbb{H} \times \mathbb{C}$ similarly to (15) and (16) respectively. Then, $\varphi \circ g' \circ \varphi^{-1} = \rho(g')$ is equivalent to $\varphi(\mu'(g')(w, z)) = \mu(\rho(g'))\varphi(w, z)$. Thus, it is sufficient to prove that $\varphi \circ (\zeta, y) = (\zeta K, \zeta \mathbf{v} + \det Ky) \circ \varphi$ for any $(\zeta, y) \in \Gamma'_r$. This is equivalent to (22) and

$$\begin{aligned} \frac{c}{\alpha - 1}(\eta\mathbf{b})\mathbf{a}' + f \left(\zeta \mathbf{c}' - \frac{\theta'}{r}y + \frac{1}{2}(\zeta \mathbf{a}')(\zeta \mathbf{b}') \right) &= -\frac{\alpha(\eta\mathbf{a})}{\alpha - 1}(\zeta K\mathbf{b}) + \zeta K\mathbf{c} - \frac{\theta}{r}((\det K)y + \mathbf{v}) \\ &\quad + \frac{1}{2}(\zeta K\mathbf{a})(\zeta K\mathbf{b}), \end{aligned}$$

for any (ζ, y) . By (22), we obtain $\frac{f}{2}(\zeta \mathbf{a}')(\zeta \mathbf{b}') = \frac{1}{2}(\zeta K\mathbf{a})(\zeta K\mathbf{b})$ and $\frac{f\theta'}{r} = \frac{\theta}{r}(\det K)$. It remains to show that

$$K\mathbf{c} - f\mathbf{c}' = \frac{c(\eta\mathbf{b})}{\alpha - 1}\mathbf{a}' + \frac{\alpha(\eta\mathbf{a})}{\alpha - 1}(K\mathbf{b}) + \frac{\theta}{r}\mathbf{v}. \quad (26)$$

By (14) and (24), (26) is equivalent to

$$\theta K \begin{pmatrix} l_2 \\ -l_1 \end{pmatrix} = c(\eta \mathbf{b}) \mathbf{a}' - \frac{f}{c}(\eta \mathbf{a}) \mathbf{b}'.$$

Hence it suffices to show that $Z(l_2, -l_1)^T = 0$ for

$$Z = \theta K - c \mathbf{a}'(b_2, -b_1) + \frac{f}{c} \mathbf{b}'(a_2, -a_1).$$

This follows from $Z \mathbf{a} = Z \mathbf{b} = 0$. \square

LEMMA 2.4. *Assume that*

$$\begin{aligned} r' &= r, \\ K N K^{-1} &= N', \\ \det K \begin{pmatrix} p' \\ q' \end{pmatrix} - K \begin{pmatrix} p \\ q \end{pmatrix} &\in r \mathbb{Z}^2 + (N' - I) \mathbb{Z}^2. \end{aligned}$$

for some $K \in GL(2, \mathbb{Z})$. Fix any $t' \in \mathbb{C}$ and eigenvectors \mathbf{a}', \mathbf{b}' of N' corresponding to $\alpha, 1/\alpha$ respectively. Then, $S_{N', p, q, r, t, \mathbf{a}, \mathbf{b}}^+$ is deformation equivalent to $S_{N', p', q', r', t', \mathbf{a}', \mathbf{b}'}^+$ or $S_{N', p', q', r', t', -\mathbf{a}', \mathbf{b}'}^+$.

Proof We use the isomorphism $\varphi : G_{N', p', q', r'}^+ \rightarrow G_{N, p, q, r}^+$ given in the proof of Lemma 2.1. By Lemma 2.2, the lift of $\varphi|_{G_{N', p', q', r'}^+}$ satisfies (24). Choose $\hat{t} \in \mathbb{C}$ and eigenvectors $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ which satisfy (22) and (23). Then $S_{N', p', q', r', \hat{t}, \hat{\mathbf{a}}, \hat{\mathbf{b}}}^+$ is biholomorphic to $S_{N, p, q, r, t, \mathbf{a}, \mathbf{b}}^+$. Note that the choice of t does not affect the deformation equivalence class. Furthermore, if $c > 0$ and $f \in \mathbb{R} \setminus \{0\}$, then $S_{N, p, q, r, 0, \mathbf{a}, \mathbf{b}}^+$ and $S_{N, p, q, r, 0, c\mathbf{a}, f\mathbf{b}}^+$ are biholomorphic by the map

$$S_{N, p, q, r, 0, \mathbf{a}, \mathbf{b}}^+ \rightarrow S_{N, p, q, r, 0, c\mathbf{a}, f\mathbf{b}}^+ : (w, z) \mapsto (cw, fz).$$

This proves the lemma. \square

Let $S_{N_i, p_i, q_i, r_i, t_i, \mathbf{a}_i, \mathbf{b}_i}^+$ be Inoue surfaces homotopy equivalent to $S_{N, p, q, r, t, \mathbf{a}, \mathbf{b}}^+$, where $i = 1, 2$. For each i , there exists $K_i \in GL(2, \mathbb{Z})$ and $\delta_i, \epsilon_i \in \{1, -1\}$ which satisfies (8) - (10) of Lemma 2.1. By Lemma 2.4, we can easily show the following corollary.

COROLLARY 2.5. *If $\det K_1 = K_2, \delta_1 = \delta_2$ and $\epsilon_1 = \epsilon_2$, then $S_{N_2, p_2, q_2, r_2, t_2, \mathbf{a}_2, \mathbf{b}_2}^+$ is deformation equivalent either to $S_{N_1, p_1, q_1, r_1, t_1, \mathbf{a}_1, \mathbf{b}_1}^+$ or to $S_{N_1, p_1, q_1, r_1, t_1, -\mathbf{a}_1, \mathbf{b}_1}^+$.*

PROPOSITION 2.6. *The number of deformation equivalence classes of Inoue surfaces of type S^+ homotopy equivalent to $S_{N, p, q, r, t, \mathbf{a}, \mathbf{b}}^+$ is at most 16.*

Proof We first construct 16 surfaces which are homotopy equivalent to $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+$.

Let $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For every $(d, \delta, \epsilon) \in \{1, -1\}^3$, we choose integers $P(d, \delta, \epsilon)$ and $Q(d, \delta, \epsilon)$ satisfying

$$\delta \begin{pmatrix} P(d, \delta, \epsilon) \\ Q(d, \delta, \epsilon) \end{pmatrix} - \epsilon J^d \begin{pmatrix} p \\ q \end{pmatrix} \in r\mathbb{Z}^2 + (J^{\frac{-d+1}{2}} N^\epsilon J^{\frac{d-1}{2}}) \mathbb{Z}^2.$$

Fix eigenvectors $\mathbf{a}(d, \delta, \epsilon)$ and $\mathbf{b}(d, \delta, \epsilon)$ of the matrix $J^{\frac{-d+1}{2}} N^\epsilon J^{\frac{d-1}{2}}$ corresponding to eigenvalues α and $\frac{1}{\alpha}$ respectively. Set

$$\begin{aligned} S_{(d,\delta,\epsilon,+)}^+ &= S_{J^{\frac{-d+1}{2}} N^\epsilon J^{\frac{d-1}{2}}, P(d,\delta,\epsilon), Q(d,\delta,\epsilon), d\delta r, \mathbf{a}(d,\delta,\epsilon), \mathbf{b}(d,\delta,\epsilon)}^+, \\ S_{(d,\delta,\epsilon,-)}^+ &= S_{J^{\frac{-d+1}{2}} N^\epsilon J^{\frac{d-1}{2}}, P(d,\delta,\epsilon), Q(d,\delta,\epsilon), d\delta r, -\mathbf{a}(d,\delta,\epsilon), \mathbf{b}(d,\delta,\epsilon)}^+. \end{aligned}$$

Now, let S' be an Inoue surface of type S^+ homotopy equivalent to $S_{N,p,q,r,t,\mathbf{a},\mathbf{b}}^+$. Write $S' = S_{N',p',q',r',t',\mathbf{a}',\mathbf{b}'}^+$. Then, there exists $K \in GL(2, \mathbb{Z})$, $\epsilon, \delta \in \{1, -1\}$ which satisfies (8) - (10). By Corollary 2.5, S' is deformation equivalent either to $S_{(\det K, \delta, \epsilon, +)}^+$ or to $S_{(\det K, \delta, \epsilon, -)}^+$. This proves the proposition. \square

3 Inoue surfaces of type S^-

We first recall the definition of Inoue surfaces of type S^- . Let $N = (n_{ij}) \in GL(2, \mathbb{Z})$ be a matrix with eigenvalues $\alpha > 1, -1/\alpha$. We choose eigenvectors $\mathbf{a} = {}^T(a_1, a_2) \in \mathbb{R}^2$, $\mathbf{b} = {}^T(b_1, b_2) \in \mathbb{R}^2$ of N corresponding to $\alpha, -1/\alpha$ respectively. Fix integers p, q and r , where $r \neq 0$. Let $\theta = \det(\mathbf{a}, \mathbf{b})$. We define $c_1, c_2 \in \mathbb{R}$ to be the solution of

$$(N + I) \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} - \frac{\theta}{r} \begin{pmatrix} p \\ q \end{pmatrix} = 0,$$

where $e_i = \frac{1}{2}n_{i1}(n_{i1} - 1)a_1b_1 + \frac{1}{2}n_{i2}(n_{i2} - 1)a_2b_2 + n_{i1}n_{i2}b_1a_2$ for $i = 1, 2$. Let $G_{N,p,q,r}^-$ be the subgroup of $\text{Aut}(\mathbb{H} \times \mathbb{C})$ generated by

$$g_0 : (w, z) \mapsto (\alpha w, z), \quad (27)$$

$$g_i : (w, z) \mapsto (z + a_i, w + b_i z + c_i), \quad (28)$$

$$g_3 : (w, z) \mapsto (z, w - \frac{\theta}{r}), \quad (29)$$

where $i = 1, 2$. The quotient surface $S_{N,p,q,r,\mathbf{a},\mathbf{b}}^- = \mathbb{H} \times \mathbb{C} / G_{N,p,q,r}^-$ is called an Inoue surface of type S^- .

By (27) - (29), we have the following relations:

$$g_0 g_1 g_0^{-1} = g_1^{n_{11}} g_2^{n_{12}} g_3^p, g_0 g_2 g_0^{-1} = g_1^{n_{21}} g_2^{n_{22}} g_3^q, g_0 g_3 g_0^{-1} = g_3^{-1}, \quad (30)$$

$$g_1 g_2 g_1^{-1} g_2^{-1} = g_3^r, g_i g_3 = g_3 g_i, \quad (31)$$

where $i = 1, 2$. As in section 2, we can show that any element $g \in G_{N,p,q,r}^-$ can be written uniquely as a product $g = \prod_{i=0}^3 g_i^{n_i}$, where n_0, n_1, n_2, n_3 are integers. Thus, $G_{N,p,q,r}^-$ has a presentation in terms of the generators g_0, g_1, g_2, g_3 , with relations (30) and (31). In this case, the center of $G_{N,p,q,r}^-$ is trivial.

Let $\Gamma_{G_{N,p,q,r}^-}$ be the subgroup of $G_{N,p,q,r}^-$ defined in (7). Since $\Gamma_{G_{N,p,q,r}^-}$ is isomorphic to $\Gamma_{G_{N,p,q,r}^+}$, properties (i), (ii), and (iii) of section 2 holds for $\Gamma_{G_{N,p,q,r}^-}$ as well. The quotient group $G_{N,p,q,r}^- / \Gamma_{G_{N,p,q,r}^-}$ is an infinite cyclic group generated by the class of g_0 .

LEMMA 3.1. *Inoue surfaces $S_{N,p,q,r,\mathbf{a},\mathbf{b}}^-$ and $S_{N',p',q',r',\mathbf{a}',\mathbf{b}'}^-$ constructed above are homotopy equivalent if and only if there exists $K \in GL(2, \mathbb{Z}), \delta \in \{1, -1\}$ such that*

$$r' = \delta \det K r, \quad (32)$$

$$N' = K N K^{-1}, \quad (33)$$

$$\delta \begin{pmatrix} p' \\ q' \end{pmatrix} - K \begin{pmatrix} p \\ q \end{pmatrix} \in r \mathbb{Z}^2 + (N' + I) \mathbb{Z}^2. \quad (34)$$

Proof The proof is similar to Lemma 2.1. Assume that $S_{N,p,q,r,\mathbf{a},\mathbf{b}}^-$ and $S_{N',p',q',r',\mathbf{a}',\mathbf{b}'}^-$ are homotopy equivalent. Then, there exists an isomorphism $\varphi : G_{N',p',q',r'}^- \rightarrow G_{N,p,q,r}^-$. The properties above implies that there exists $\epsilon, \delta \in \{-1, 1\}$ and integers $k_{ij} (i = 1, 2, j = 1, 2, 3)$ such that

$$\begin{aligned} \varphi(g'_0) &= g_0^\epsilon g_1^{k_{01}} g_2^{k_{02}} g_3^{k_{03}}, \\ \varphi(g'_i) &= g_1^{k_{i1}} g_2^{k_{i2}} g_3^{k_{i3}}. \end{aligned}$$

Suppose $\epsilon = 1$. We have (32), (33) and (34) by a similar calculation in Lemma 2.1. If $\epsilon = -1$, it follows similarly that $K N^{-1} K^{-1} = N'$. However this is a contradiction, since N' must have an eigenvalue $0 < \alpha < 1$.

Next we show the converse. We construct an isomorphism $\varphi : G_{N',p',q',r'}^- \rightarrow G_{N,p,q,r}^-$ similarly to Lemma 2.1. Hence, $S_{N,p,q,r,\mathbf{a},\mathbf{b}}^-$ and $S_{N',p',q',r',\mathbf{a}',\mathbf{b}'}^-$ are diffeomorphic to each other since they are both solvmanifolds [5]. \square

LEMMA 3.2. Assume that $r = r'$ and that there exists $K \in GL(2, \mathbb{Z})$, $c > 0$, $u, v \in \mathbb{Z}$ such that

$$\eta \in \mathbb{Z}^2, K N K^{-1} = N', c \mathbf{a}' = K \mathbf{a}, f \mathbf{b}' = c K \mathbf{b},$$

where $f = \frac{\det K \theta}{\theta'}$ and $\eta = (l_1, l_2) \in \mathbb{Z}^2$ is a solution of

$$(N' + I) \begin{pmatrix} u + \frac{r}{2} k_{11} k_{12} \\ v + \frac{r}{2} k_{21} k_{22} \end{pmatrix} = K \mathbf{p} - (\det K) \mathbf{p}' - K r \begin{pmatrix} -l_2 \\ l_1 \end{pmatrix}.$$

Then $S_{N, p, q, r, \mathbf{a}, \mathbf{b}}^-$ and $S_{N', p', q', r', \mathbf{a}', \mathbf{b}'}^-$ are biholomorphic.

To prove this lemma, we apply Γ_r for Inoue surfaces of type S^- . By replacing (8.3) of [4] by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -1 \end{pmatrix}$, equation (8.4) of [4] becomes

$$(N + I) \mathbf{c} = \frac{\theta}{r} \mathbf{p}, \quad (35)$$

where \mathbf{c} and \mathbf{p} are column vectors defined in section 2. We define the homomorphism $\mu : \Gamma_{G_{N, p, q, r}^-} \rightarrow \Gamma_r$ by (13), and the action Γ_r on $\mathbb{H} \times \mathbb{C}$ by (15). We can show the following lemma similarly to Lemma 2.2:

LEMMA 3.3. Assume that $r = r'$. Let $\rho : G_{N', p', q', r'}^- \rightarrow G_{N, p, q, r}^-$ be an isomorphism such that $\rho(g'_0) = g_0 g_1^{l_1} g_2^{l_2}$ for some integers l_1, l_2 . Denote by $(K, \mathbf{v}) \in GL(2, \mathbb{Z}) \times \mathbb{Z}[r/2]^2$, the lift of $\rho|_{\Gamma_{G_{N', p', q', r'}^-}}$. Then (K, \mathbf{v}) satisfies the following conditions:

$$K N K^{-1} = N', \quad (N' + I) \mathbf{v} = K \mathbf{p} - (\det K) \mathbf{p}' - K r \begin{pmatrix} -l_2 \\ l_1 \end{pmatrix}, \quad (36)$$

$$v_i - (r/2) k_{i1} k_{i2} \in \mathbb{Z}, \quad (37)$$

where $\mathbf{v} = {}^T(v_1, v_2)$. Conversely, assume that $(K, \mathbf{v}) \in GL(2, \mathbb{Z}) \times \mathbb{Z}[r/2]^2$ satisfies (36) and (37) for some integers l_1 and l_2 . Let $\varphi : \Gamma_{G_{N', p', q', r'}^-} \rightarrow \Gamma_{G_{N, p, q, r}^-}$ be an isomorphism whose lift corresponds to (K, \mathbf{v}) . Then, the map

$$\rho : G_{N', p', q', r'}^+ \rightarrow G_{N, p, q, r}^+ : g_0'^{l_0} \gamma \mapsto (g_0 g_1^{l_1} g_2^{l_2})^{l_0} \varphi(\gamma),$$

is an isomorphism, where $\gamma \in \Gamma_{G_{N', p', q', r'}^-}$.

Proof of Lemma 3.2 Let $\varphi : \mathbb{H} \times \mathbb{C} \rightarrow \mathbb{H} \times \mathbb{C} : (w, z) \mapsto (cw + d, ew + fz + g)$, where

$$\begin{aligned} d &= \frac{\alpha}{1-\alpha} \eta \mathbf{a}, & e &= c \frac{\eta \mathbf{b}}{\alpha+1}, \\ g &= \frac{1}{2}(-d\eta \mathbf{b} - \eta \mathbf{c} + \frac{\theta}{2} l_1 l_2) + \frac{1}{4}(\eta \mathbf{a})(\eta \mathbf{b}). \end{aligned}$$

Set $\mathbf{v} = \begin{pmatrix} u + \frac{r}{2} k_{11} k_{12} \\ v + \frac{r}{2} k_{21} k_{22} \end{pmatrix}$. It suffices to show that $\varphi \circ g'_0 \circ \varphi^{-1} = g_0 g_1^{l_1} g_2^{l_2}$ and $\varphi \circ (\zeta, y) = (\zeta K, \zeta \mathbf{v} + \det K y) \circ \varphi$ for any $(\zeta, y) \in \Gamma_r$.

This can be proved similarly to Lemma 2.3 by using (35) and (36). \square

COROLLARY 3.4. *Assume that there exists $K \in GL(2, \mathbb{Z})$ such that*

$$\begin{aligned} r' &= r, K N K^{-1} = N', \\ \det K \begin{pmatrix} p' \\ q' \end{pmatrix} - K \begin{pmatrix} p \\ q \end{pmatrix} &\in r\mathbb{Z}^2 + (N' + I)\mathbb{Z}^2. \end{aligned}$$

Fix any eigenvectors \mathbf{a}', \mathbf{b}' of N' corresponding to $\alpha, -1/\alpha$ respectively. Then, $S_{N', p, q, r, \mathbf{a}, \mathbf{b}}^-$ is deformation equivalent either to $S_{N', p', q', r', \mathbf{a}', \mathbf{b}'}^-$ or $S_{N', p', q', r', -\mathbf{a}', \mathbf{b}'}^-$.

PROPOSITION 3.5. *The number of deformation equivalence classes of Inoue Surfaces of type S^- homotopy equivalent to $S_{N', p, q, r, \mathbf{a}, \mathbf{b}}^-$ is less than or equal to 8.*

The statements above can be shown similarly to section 2.

4 Deformation equivalence classes of Inoue surfaces of type S^0, S^+ or S^-

In this section, we first review the classification of compact surfaces with $b_1 = 1, b_2 = 0$. Then, we show that the number of deformation equivalence classes of surfaces homotopy equivalent to an Inoue surface of type S^0, S^+ , or S^- is at most 16.

PROPOSITION 4.1. *Let S be a compact surface with $b_1(S) = 1$ and $b_2(S) = 0$. Then S is either an elliptic surface, a Hopf surface, or an Inoue surface of type S^0, S^+ or S^- .*

Proof Let $\kappa(S)$ be the Kodaira dimension of S . Since $b_1(S)$ is odd, $\kappa(S)$ is either $-\infty, 0$ or 1 . If $\kappa(S) = 1$, then S is elliptic [1]. If $\kappa(S) = 0$, S must be a secondary Kodaira Surface, which are elliptic [1]. By Bogomolov [2] and Teleman [10], if $\kappa(S) = -\infty$, S is either a Hopf surface or an Inoue surface of type S^0, S^+ , or S^- . \square

By [6] and the results of sections 1 and 2, we see that $G_M, G_{N,p,q,r}^+, G_{N,p,q,r}^-$ has the following properties:

G_M	trivial center	Γ_{G_M}	abelian
$G_{N,p,q,r}^+$	infinite-cyclic center	$\Gamma_{G_{N,p,q,r}^+}$	non-abelian
$G_{N,p,q,r}^-$	trivial center	$\Gamma_{G_{N,p,q,r}^-}$	non-abelian

Table 4.2

Let S be an Inoue surface of type S^0, S^+ , or S^- , and S' be an elliptic surface with $b_1(S') = 1, b_2(S') = 0$. By the proof of [3 Chapter II Theorem 7.16], if $\pi_1(S')$ is non-abelian, then the center of $\pi_1(S')$ is isomorphic to either $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z} (n \geq 1)$ or \mathbb{Z}^2 . Hence, $\pi_1(S)$ is not isomorphic to $\pi_1(S')$. Recall that a Hopf surface is by definition, a surface with universal cover $\mathbb{C}^2 \setminus \{0\}$. Thus, S cannot deform to elliptic surfaces or to Hopf surfaces. Furthermore, Proposition 4.1 and Table 4.2 implies that any surface deformation equivalent to S must deform through Inoue surfaces of the same type. Therefore, the number of deformation equivalence classes of surfaces homotopy equivalent to an Inoue surface with $b_2 = 0$ is at most 16.

5 Elliptic Surfaces and Hopf Surfaces

To complete the proof of our main theorem, we must consider the case when S is a Hopf surface or an elliptic surface with $b_1(S) = 1$ and $b_2(S) = 0$. According to [3. Chapter I Lemma 7.20], if S is a Hopf surface with a non-abelian fundamental group, then S is elliptic. It is also known that if S is an elliptic surface whose fundamental group is abelian, then S is a Hopf Surface [3. Chapter II Proposition 7.5]. Hence, we may assume that S is either an elliptic surface whose fundamental group is non-abelian, or a Hopf surface with an abelian fundamental group.

PROPOSITION 5.1. *Let S be an elliptic surface with $b_1(S) = 1, b_2(S) = 0$. Assume that $\pi_1(S)$ is non-abelian. Then the number of deformation equivalence classes of surfaces homotopy equivalent to S is at most two.*

Proof According to [9], S is obtained by performing logarithmic transformation finitely times over $\mathbb{CP}^1 \times E$, where E is a general elliptic curve. Since the base \mathbb{CP}^1 is simply connected, S is an elliptic surface with cyclic monodromy. Let S' be a surface homotopy equivalent to S . By [3. Chapter II Corollary 7.17], S' is deformation equivalent either to S or to S^{conj} where S^{conj} is the conjugate complex manifold of S (See [3. Chapter II Definition 7.13] for the definition of S^{conj}). \square

PROPOSITION 5.2. *Let S be a Hopf surface whose fundamental group is abelian. Then the number of deformation equivalence classes of surfaces homotopy equivalent to S is finite.*

Proof We refer to [3. Chapter I Section 1.7.6] for details. If $\pi_1(S) \cong \mathbb{Z}$, the number of deformation equivalence classes of surfaces homotopy equivalent to S is one since Hopf surfaces with an infinite-cyclic fundamental group are all deformation equivalent. If $\pi_1(S) \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$ ($n > 1$), S is diffeomorphic to $S_1 \times L(n, q)$ for some $q \in (\mathbb{Z}/n\mathbb{Z})^*$, where $L(n, q)$ is a lens space. Since the choice of q is finite, the number of deformation equivalence classes of surfaces homotopy equivalent to S must be finite. \square

Remark By the discussion of [3], the number of deformation equivalence classes of surfaces diffeomorphic to S is at most two.

This completes the proof of our main theorem.

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